Exercice 7, feuille TD3

① $f_{X,Y}(x,y) = \frac{1}{4}(x+y)e^{-y}\,\mathbb{1}_{[0,2]}(x)\,\mathbb{1}_{\mathbb{R}_+}(y)$.

On a $f_{X,Y}(x,y) \geq 0 \ \forall x,y \in \mathbb{R}$. Puis,

$$\int_{\mathbb{R}^2} \underbrace{f_{X,Y}(x,y)}_{\geq 0}\,dx\,dy \overset{\text{Fubini}}{=} \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f_{X,Y}(x,y)\,dx\right)dy$$

$$= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \frac{1}{4}(x+y)e^{-y}\,\mathbb{1}_{[0,2]}(x)\,\mathbb{1}_{\mathbb{R}_+}(y)\,dx\right)dy$$

$$= \frac{1}{4}\int_{\mathbb{R}} e^{-y}\,\mathbb{1}_{\mathbb{R}_+}(y)\left(\int_{\mathbb{R}}(x+y)\,\mathbb{1}_{[0,2]}(x)\,dx\right)dy$$

$$= \frac{1}{4}\int_0^{+\infty} e^{-y}\left(\int_0^2 (x+y)\,dx\right)dy = \frac{1}{4}\int_0^{+\infty} e^{-y}\left[\frac{x^2}{2}+xy\right]_{x=0}^{x=2} dy$$

$$= \frac{1}{4}\int_0^{+\infty} \underbrace{e^{-y}}_{=v'}\underbrace{(2+2y)}_{=u}\,dy \overset{\text{IPP}}{=} \frac{1}{4}\left(\left[(2+2y)(-e^{-y})\right]_0^{+\infty} - \int_0^{+\infty} 2\times(-e^{-y})\,dy\right)$$

$$= \frac{1}{4}\left(2 + 2\int_0^{+\infty} e^{-y}\,dy\right) = \frac{1}{4}\left(2 + 2\left[-e^{-y}\right]_0^{+\infty}\right) = \frac{2+2}{4} = 1$$

Donc $f_{X,Y}$ est une densité.

(2) $$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy = \int_{\mathbb{R}} \frac{1}{4}(x+y)e^{-y}\,\mathbb{1}_{[0,2]}(x)\,\mathbb{1}_{\mathbb{R}_+}(y)\,dy$$

$$= \frac{1}{4}\mathbb{1}_{[0,2]}(x)\int_{\mathbb{R}} (x+y)e^{-y}\,\mathbb{1}_{\mathbb{R}_+}(y)\,dy = \frac{1}{4}\mathbb{1}_{[0,2]}(x)\int_0^{+\infty} \underbrace{(x+y)}_{=u}\underbrace{e^{-y}}_{=v'}\,dy$$

IPP

$$= \frac{1}{4}\mathbb{1}_{[0,2]}(x)\left(\underbrace{\left[(x+y)(-e^{-y})\right]_0^{+\infty}}_{=x} - \int_0^{+\infty} 1\times(-e^{-y})\,dy\right)$$

$$= \frac{1}{4}\mathbb{1}_{[0,2]}(x)\left(x + \int_0^{+\infty} e^{-y}\,dy\right) = \frac{1}{4}\mathbb{1}_{[0,2]}(x)\left(x + \left[-e^{-y}\right]_0^{+\infty}\right)$$

$$= \frac{1}{4}\mathbb{1}_{[0,2]}(x)(x+1)$$

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dx = \int_{\mathbb{R}} \frac{1}{4}(x+y)e^{-y}\mathbb{1}_{[0,2]}(x)\,\mathbb{1}_{\mathbb{R}_+}(y)\,dx$$

$$= \frac{1}{4}e^{-y}\mathbb{1}_{\mathbb{R}_+}(y)\int_{\mathbb{R}}(x+y)\mathbb{1}_{[0,2]}(x)\,dx = \frac{1}{4}e^{-y}\mathbb{1}_{\mathbb{R}_+}(y)\int_0^2(x+y)\,dx$$

$$= \frac{1}{4}e^{-y}\mathbb{1}_{\mathbb{R}_+}(y)\left[\frac{x^2}{2}+xy\right]_0^2 = \frac{1}{4}e^{-y}\mathbb{1}_{\mathbb{R}_+}(y)(2+2y)$$

③ $F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & \text{si } x<0 \\ 1 & \text{si } x>2 \end{cases}$ $\quad f_X(x) = \frac{x+1}{4}\mathbb{1}_{[0,2]}(x)$

Puis, si $x \in [0,2]$:

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x}\frac{t+1}{4}\mathbb{1}_{[0,2]}(t)\,dt = \int_0^x \frac{t+1}{4}\,dt \text{ car } x\in[0,2].$$

$$= \frac{1}{4}\left[\frac{t^2}{2}+t\right]_0^x = \frac{1}{4}\left(\frac{x^2}{2}+x\right)$$

On cherche alors $M \in \mathbb{R}$ t.q. $F_X(M) = \frac{1}{2}$. On a $M \in [0,2]$ (car sinon on aurait $F_X(M) = 0$ ou 1), donc $F_X(M) = \frac{1}{2} \iff \frac{1}{4}\left(\frac{M^2}{2}+M\right) = \frac{1}{2}$

$\iff \frac{M^2}{2} + M = 2 \iff M^2 + 2M - 4 = 0 \qquad \Delta = 4+16 = 20$ donc

$M = \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$. Comme $M \geq 0$, $\boxed{M = -1+\sqrt{5}}$

Pour Y : $f_Y(y) = \frac{1+y}{2} e^{-y} \mathbb{1}_{\mathbb{R}_+}(y)$. $F_Y(y) = \mathbb{P}(Y \leq y) = 0$ si $y < 0$

Puis, si $y \geq 0$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \int_{-\infty}^{y} f_Y(t)\,dt = \int_{-\infty}^{y} \frac{1+t}{2} e^{-t} \mathbb{1}_{\mathbb{R}_+}(t)\,dt$$

$$= \int_0^y \underbrace{\frac{1+t}{2}}_{=u} \underbrace{e^{-t}}_{=v'}\,dt \overset{\text{IPP}}{=} \left[\frac{1+t}{2}(-e^{-t})\right]_0^y - \int_0^y \frac{1}{2}(-e^{-t})\,dt$$

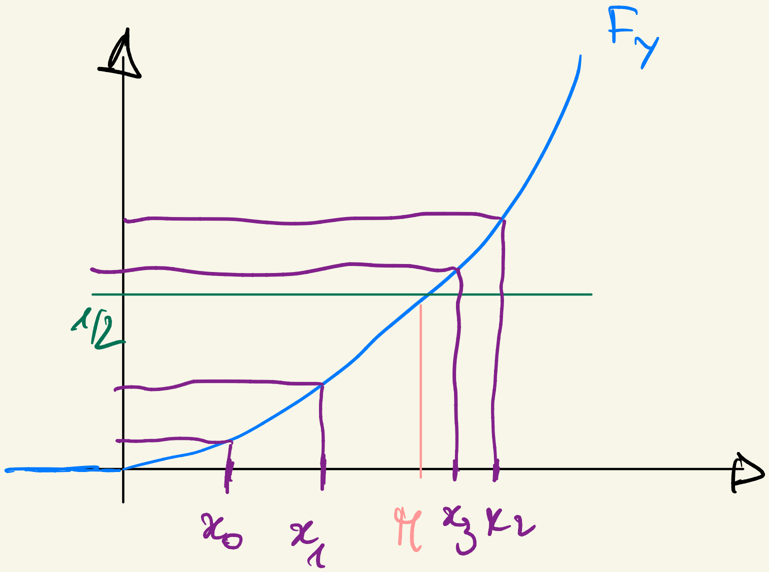
$$= \frac{1+y}{2}(-e^{-y}) - \frac{1}{2}(-1) + \frac{1}{2}\int_0^y e^{-t}\,dt = -\frac{1+y}{2}e^{-y} + \frac{1}{2} + \frac{1}{2}\left[-e^{-t}\right]_0^y$$

$$= -\frac{1+y}{2}e^{-y} + \frac{1}{2} + \frac{1}{2}\left(-e^{-y} - (-1)\right) = -\frac{1+y}{2}e^{-y} + 1 - \frac{1}{2}e^{-y}$$

$$= -e^{-y} - \frac{y}{2}e^{-y} + 1$$

On cherche $M \in \mathbb{R}$ t.q. $F_Y(M) = \frac{1}{2}$. On a donc $M \geq 0$ (sinon $F_Y(M) = 0$).

Alors $F_Y(M) = \frac{1}{2} \iff -e^{-M} - \frac{M}{2}e^{-M} + 1 = \frac{1}{2}$ d'où $M \simeq 1{,}146$



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$$\mathbb{E}(X) = \int_{\mathbb{R}} x \, \frac{x+1}{4} \, \mathbb{1}_{[0,2]}(x) \, dx = \frac{1}{4} \int_0^2 x(x+1) \, dx = \frac{1}{4} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_0^2$$

$$= \frac{1}{4} \left(\frac{8}{3} + 2 \right) = \frac{1}{4} \left(\frac{8+6}{3} \right) = \frac{14}{12} = \frac{7}{6}$$

$$\mathbb{E}(Y) = \int_{\mathbb{R}_+} y \, \frac{1+y}{2} e^{-y} \, dy = \frac{1}{2} \underbrace{\int_0^{+\infty} y e^{-y} \, dy}_{=1} + \frac{1}{2} \underbrace{\int_0^{+\infty} y^2 e^{-y} \, dy}_{[y^2(-e^{-y})]_0^{+\infty} - 2\int_0^{+\infty} y(-e^{-y}) \, dy}$$

$$= 2 \int_0^{+\infty} y e^{-y} \, dy = 2$$

$$= \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$$